Pioneer Journal of Mathematics and Mathematical Sciences
Volume 25, Number 1, 2019, Pages 1-7
This paper is available online on February 25, 2019 at http://www.pspchv.com/content_PJMMS.html

THE DIRICHLET PROBLEM ON A RIEMANNIAN VARIETY

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Abstract

The problem of Dirichlet is often partially studied in analysis, the purpose of this article is to bring this problem back to Riemannian geometry and make it a wider field of resolution on Riemannian varieties. This paper reports on our joint work [6].

1. Introduction

Let M be a Riemannian variety, and $\Omega \subset M$ consider the problem of minimising

$$\frac{1}{2}\int_{\Omega} |\nabla u|^2 dx,$$

where Ω is a bounded, regular, simply connected domain in M and $u : \Omega \to \mathbb{S}^1$ is a unit vector field with a Dirichlet boundary condition $g : \partial\Omega \to \mathbb{S}^1$, which is, say, C^1.

Received February 6, 2019

2010 Mathematics Subject Classification: 58E20, 49J10, 82D55.

Keywords and phrases: minimisation, renormalized energy, degree.

Two cases occur:

(1) $\deg(g) = 0$, here $\deg(g)$ is the winding number of the map g. In this case, we may write $g = e^{i\phi}$, where $\phi : \partial\Omega \to \mathbb{R}$. If we call $\tilde{\phi}$ the harmonic extension of ϕ to Ω, $e^{i\tilde{\phi}}$ is the solution to our minimisation problem.

(2) $\deg(g) \neq 0$. Here we know that no continuous extension of g inside Ω exists. Worse, no finite energy extension exists. An intuitive justification for this may be that according to a results of Baird and Kamissoko [1], Schoen and Uhlenbeck [5], smooth maps are dense in the Sobolev space $H^1(\Omega, \mathbb{S}^1)$, thus a finite energy extension would be close to a continuous one, which does not exist. However, using results of Bethuel et al. [2], a variational problem can still be worked out.

Consider the set $\mathcal{C}$ of maps from Ω to $\mathbb{S}^1$ that are sufficiently smooth outside a unite number of points in Ω, and such that if $u \in \mathcal{C}$ and p is a singularity of u, $x \to u(x) - e^{i\theta_p \frac{x-p}{|x-p|}}$ is smooth in a neighborhood of p, for some number θ_p. For any $u \in \mathcal{C}$, $\deg(u, \partial\Omega) = n$, where n is the number of singular points of u. Moreover, writhing $s(u) = \{p_1, ..., p_n\}$ and $D(p, r)$ the disc of radius r centered at p, the limit

$$\lim_{\varepsilon \to 0} \frac{1}{2} \int_{\Omega \setminus U_{i=1}^{n} D(p_i, \varepsilon)} |\nabla u|^2 dx - \pi n \log \frac{1}{\varepsilon},$$

exists, we call it $E(u)$. Note that

$$\pi \log \frac{1}{\varepsilon} = \frac{1}{2} \int_{D(0, 1) \setminus D(0, \varepsilon)} \left| \nabla \frac{x}{|x|} \right|^2 dx,$$

which explains the above expansion of the energy.

The aforementioned results of [2] are that for any given map $g : \Omega \to \mathbb{S}^1$ with degree n and any n-tuple of distinct points in Ω, $\{p_1, ..., p_n\}$, there exists a unique $u \in \mathcal{C}$ agreeing with g on the boundary, with singular set $s(u) = \{p_1, ..., p_n\}$, and harmonic in the classical sense outside $s(u)$. Moreover,

$$E(u) = \min_{v \in \mathcal{C} v = g \ on \ \partial\Omega s(v) = \{p_1, \dots, p_n\}} E(v),$$

and $E(u)$ can be computed as a function of g, $\{p_1, \dots, p_n\}$ called the renormalized energy, $W(g, \{p_i\})$. This function is computed in [2], a nice expression was given by Leffter and Radulescu [3] in the case where Ω is the unit disc and g is the map $e^{i\theta} \to e^{in\theta}$. Also, using a remark of C. Ragazzo (see [2]), the following expression for W can given.

For $x \in \partial\Omega$ let the real function $f(x) = g(x) \wedge g_\tau(x)$, where the subscript τ denotes the derivative of in the direction tangent to $\partial\Omega$ (if $g = e^{i\phi}$, then $f = \phi_\tau$). Call $G_g : \Omega \times \Omega \to \mathbb{R}$ the solution to the following problem:

$$\begin{cases} \Delta_x G_g(x, p) = 2\pi\delta_p & x, p \in \Omega \\ (G_g)_{v_x}(x, p) = f(x) & p \in \Omega, \quad x \in \partial\Omega \\ \int_{\partial\Omega} G_g(x, p) f(x) dl(x) = 0 & p \in \Omega. \end{cases} \tag{1}$$

It is a fact that $\gamma_g(x, p) = G_g(x, p) - \log|x - p|$ is a smooth function on $\Omega \times \Omega$, and the renormalized energy is

$$W(g, p_1, \dots, p_n) = -\pi \sum_{i \neq j} G_g(p_i, p_j) - \sum_i \gamma_g(p_i, p_i). \tag{1}$$

2. Results

Now, our concern is the following: Let $\left\{g_n : \partial\Omega \to \mathbb{S}^1\right\}_n$ be a sequence of boundary maps such that

$$\deg(g_n) = n,$$

and suppose that $S_n = \{p_{1,n}, \dots, p_{n,n}\}$ minimises $W(g_n, .)$, over all n-tuples of distinct points in Ω. What does S_n look like, asymptotically? We give a result, a sketch of its proof, and in the next section, a list of open problems.

Theorem 1. *Suppose* $g : \partial\Omega \to \mathbb{S}^1$ *has degree* 1, *and for any* $n \in \mathbb{N}$ *let* $g_n = g^n$, *and* $\{p_{1,n}, ..., p_{n,n}\}$ *be a minimiser of* $W(g_n, .)$. *Then the sequence of probability measures*

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{p_{i,\,n}}$$

converges weakly to a probability measure $\bar{\mu}$ *with the following properties*:

1. $\bar{\mu} \le \frac{1}{2\pi} f^+ dl$, *where* $f = g \wedge g_\tau$, *the superscript '+' denotes the positive part, and dl is the line element of* $\partial\Omega$.

2. *When* $f \ge 0$ *this corresponds to the case of a monotonous* g – *then necessarily* $\bar{\mu} = fdl$ *since*

$$\int_{\partial\Omega} fdl = \int_{\partial\Omega} g \wedge g_\tau dl = 2\pi \deg(g).$$

Sketch of the proof. We may write (1) in terms of the measures μ_n as

$$W_n(\mu_n, g) = -\pi n^2 \iint_{\Omega\times\Omega\setminus\Gamma} G_g(x, y) d\mu_n(x) d\mu_n(y) - \pi n \int_\Gamma \gamma_g(x, x) d\mu_n(x),$$

where Γ is the diagonal of $\Omega \times \Omega$.

Guessing what the limit of the μ_n's should be is now quite easy. It will be the minimiser of a functional defined on the set of all probability measures, a functional which is the limit of the one above as n goes to infinity, and whose expression we have no difficulty guessing:

$$I_g(\mu) := -\iint_{\bar{\Omega}\times\bar{\Omega}} G_g(x, y) d\mu(x) d\mu(y).$$

All we have done is keep the n^2 term in W_n, integrating it over $\bar{\Omega}\times\bar{\Omega}$ instead of $\bar{\Omega}\times\bar{\Omega}\setminus\Gamma$ in order to have a well-posed minimisation problem.

There are now two steps in the proof. First, we need to prove that the functional I_g is indeed the (gamma) limit of the functionals W_n. More precisely, we mean that the minimisers of W_n over all probability measures that are average of n Dirac masses converge to minimisers of I_g over the set of probability measures supported in $\overline{\Omega}$. There are no difficulties in this step worth mentioning.

The second step is to characterize the minimisers of I_g. Although this is technically easier than the first step, we give an idea of how this goes. First, it is useful to deduce a different expression for the energy I_g. Denote by $G_0(x, y)$ the solution of

$$\begin{cases} \Delta_x G_0(x, y) = 2\pi\delta_y & \textit{in } \Omega \\ G_{0\nu_x}(x, y) = \text{Constant} & \textit{for } x \in \partial\Omega \\ \int_{\partial\Omega} G_0(x, y)dx = 0 & \textit{for every } y \in \Omega. \end{cases}$$

It is easy to check, using Green's identity, that

$$G_g(x, y) = G_0(x, y) - \frac{1}{2\pi}\int_{\partial\Omega}(G_0(u, x) + G_0(u, y))g(u)du$$

$$+ \frac{1}{4\pi^2}\iint_{\partial\Omega\times\partial\Omega} G_0(u, v)g(u)g(v)dudv. \tag{2}$$

Replacing $G_g(x, y)$ in the expression for I_g yields $I_g(\mu) = I(\mu - \mu_g)$, where I is the new functional

$$I(\nu) = -\iint_{\overline{\Omega}\times\overline{\Omega}} G_0(x, y)d\nu(x)d\nu(y),$$

and where μ_g is the measure in $\overline{\Omega}$ given by

$$\mu_g = \frac{1}{2\pi} g \wedge g_\tau dl.$$

Then, it is probably well-known and, in any case, easy to prove that $I(.)$ is a positive strictly convex function over a suitable subset of the set of measures with zero mass. If μ_g is a positive measure, it is clear then that $\overline{\mu} = \mu_g$ is the unique minimiser of $I(. - \mu_g)$ over the set of probability measures supported in $\overline{\Omega}$.

When μ_g is not positive, it obviously cannot be the minimiser, since it is not a probability measure and an additional argument must be used.

3. Three Open Problems

When trying to improve a Theorem, one usually tries to either weaken the hypothesis, strengthen the conclusion, or a combination of both. Here is a list of possibilities:

1. Take any sequence of boundary maps $g_n : \partial\Omega \to \mathbb{S}^1$ such that

$$\deg(g_n) = n.$$

For such a sequence, there is a corresponding sequence of probability measures μ_n that minimise the renormalized energy. What are the accumulation points (in the weak topology) of such a sequence of measures? It should be true that they are all measures supported in $\partial\Omega$.

2. If the case we described, i.e., $g_n = g^n$ - or more generally, if $\frac{1}{n} g_n \wedge (g_n)_\tau$ is bounded on $\partial\Omega$ independently of n- can one prove that if $\{p_{1,1}, ..., p_{1,n}\}$ is a minimiser of $W(g_n, .)$, then

$$d(p_{i,n}, \partial\Omega) \leq \frac{C}{n}$$

for every $i \leq n$, for some constant C independent of n?

3. This one is a mind-teaser: If Ω is the unit disc and g_n is the map $e^{i\theta} \to e^{in\theta}$, prove that $\{p_{1,1}, ..., p_{1,n}\}$ are located at the vertices of a regular n-gon centerd at 0. As we said earlier, there is a nice expression for W in this case, computed in [5],

$$W(p_1, ..., p_n) = -\pi \sum_{i \neq j} \log |p_i - p_j| \left|1 - p_i \bar{p}_j\right| - \pi \sum_i \log \left|1 - |p_i|^2\right|.$$

To our knowledge, the symmetry of minimising configurations is open for n as small as 3.

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